Singularities, Groups, McKay Correspondence

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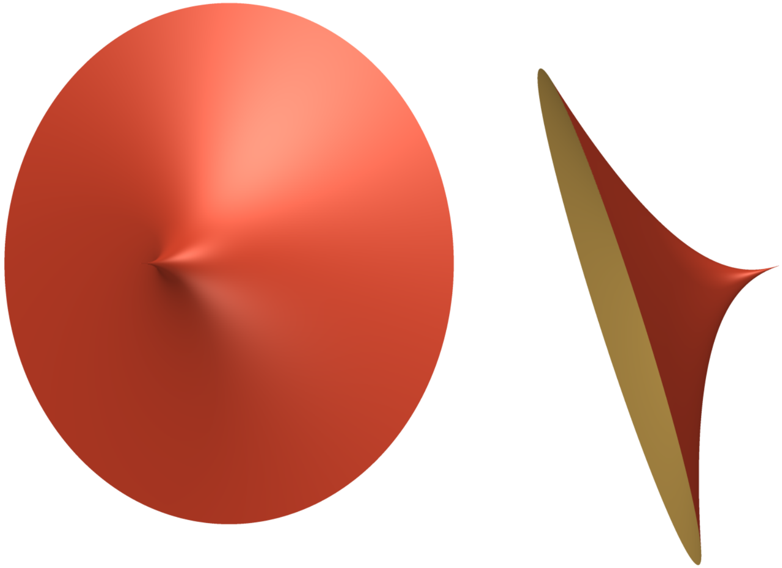
Hypersurface - Example

Consider the polynomial $f := f(x, y, z) = x^2 + y^2 + z^5 \in \mathbb{C}[x, y, z]$.

The hypersurface defined by f is

$$H := \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^5 = 0\}$$

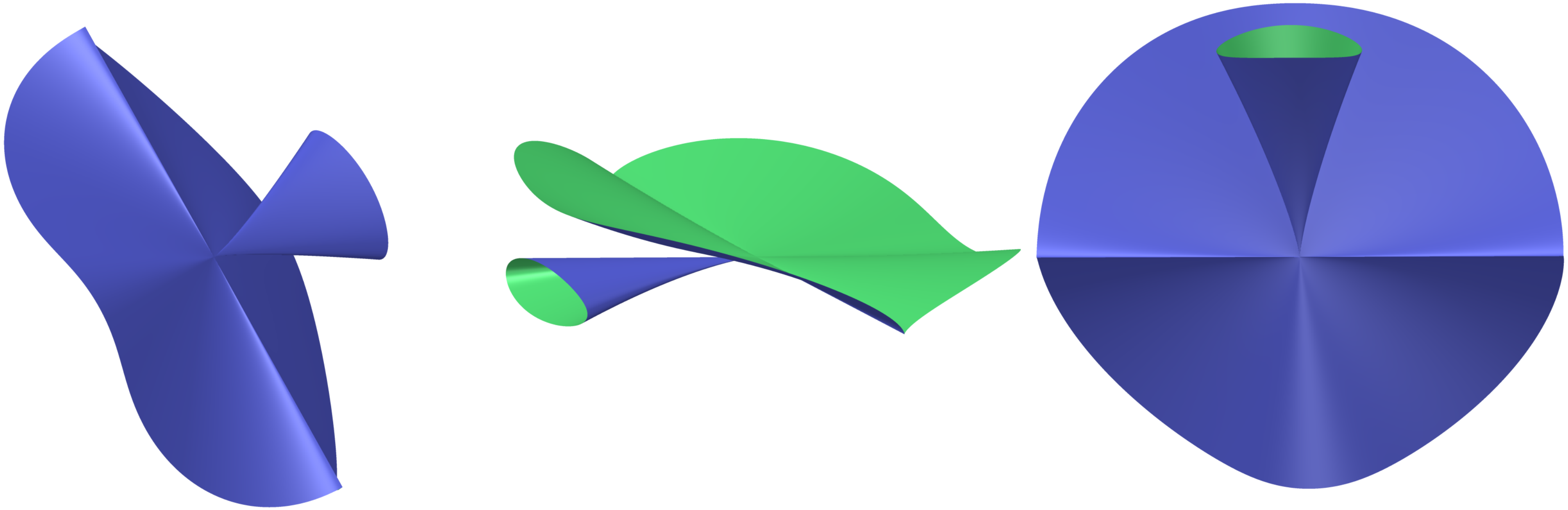


Hypersurface - Example

Consider the polynomial $f := f(x, y, z) = z^3 + x^2y^2 + y^3z \in \mathbb{C}[x, y, z]$.

The hypersurface defined by f is

$$H := \{z^3 + x^2y^2 + y^3z = 0\} \subset \mathbb{C}^3$$



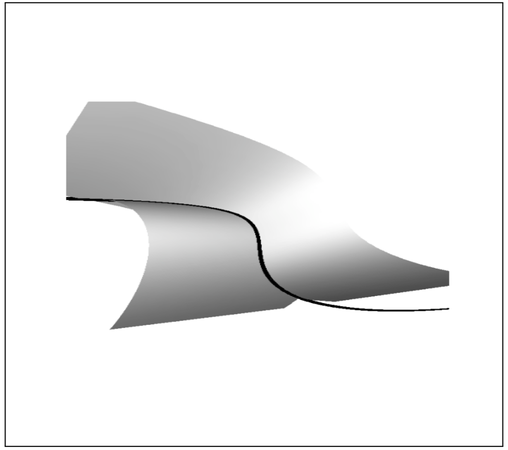
Non-Hypersurface - Example

Consider the polynomials $f_1 := y - x^2, f_2 := z - x^3 \in \mathbb{C}[x, y, z]$.

The variety defined by f_1 and f_2 is

$$V := \{\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{C}^3 \mid f_1(\mathbf{c}) = f_2(\mathbf{c}) = 0\}\{y - x^2 = 0, z - x^3 = 0\} \subset \mathbb{C}^3$$

The intersection of two hypersurfaces $y - x^2 = 0$, $z - x^3 = 0$ is the curve



Non-Hypersurface - Example

Consider the polynomial $f_1, f_2, f_3 \in \mathbb{C}[x, y, z, w]$.

$$f_1(x, y, z, w) = z^2 - yw$$

$$f_2(x, y, z, w) = zw + y^2z - x^2y$$

$$f_3(x, y, z, w) = w^2 + y^2w - x^2$$

The variety defined by f_1,f_2 and f_3 is

$V := \{\mathbf{c} = (c_1, c_2, c_3, c_4) \in \mathbb{C}^4 \mid f_1(\mathbf{c}) = f_2(\mathbf{c}) = f_3(\mathbf{c}) = 0\} \subset \mathbb{C}^4$.

Tangent space to an hypersurface - General theory

Definiton

$$H := \{\mathbf{p} = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n \mid f(a_1, a_2, \ldots, a_n) = 0\}$$

The tangent space at a point $p \in H$ is

$$T_{\mathbf{p}}H := \{(x_1, \cdots, x_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{p})(x_i - a_i) = 0\}$$

Tangent space to an hypersurface - General theory

Example

Consider the hypersurface $H : z - x^2 - y^2 = 0$ in $\mathbb{C}^3$.

For any point $\mathbf{p} = (a, b, c) \in V$, the tangent space to V at $\mathbf{p}$

$$(-2x)\mid_{(a,b,c)} (x - a) + (-2y)\mid_{(a,b,c)} (y - b) + (1)\mid_{(a,b,c)} (z - c) = 0$$

$$-2a(x - a) - 2b(y - b) + (z - c) = 0$$

When $\mathbf{p} = (0, 0, 0) \in V$, the tangent space is $z = 0$.

When $\mathbf{p} = (1, 0, 0) \in V$, the tangent space is $-2(x - 1) + (z - 1) = 0$.

Tangent space of a non-hypersurface - General theory

Example

Consider the algebraic variety $V := \{(x, y, z) \in \mathbb{C}^3 \mid y - x^2 = 0, z - x^3 = 0\} \subset \mathbb{C}^3$.

For any point $\mathbf{p} = (a, b, c) \in V$, the tangent space to V at $\mathbf{p}$ is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x}(p) & \frac{\partial f_1}{\partial y}(p) & \frac{\partial f_1}{\partial z}(p) \\ \frac{\partial f_2}{\partial x}(p) & \frac{\partial f_2}{\partial y}(p) & \frac{\partial f_2}{\partial z}(p) \end{pmatrix} \cdot \begin{pmatrix} x - a \\ y - b \\ z - c \end{pmatrix} = 0$$

For $\mathbf{p} = (0, 0, 0) \in V$, the tangent space to V at $\mathbf{p}$ is $y = z = 0$. .

Tangent space of an hypersurface - General theory

Example

Consider the hypersurface $V := x^2 + y^2 + z^5 = 0$.

For any point $\mathbf{p} = (a, b, c) \in V$, the tangent space to V at $\mathbf{p}$

$$(2x)\mid_{(a,b,c)} (x - a) + (2y)\mid_{(a,b,c)} (y - b) + (5z^4)\mid_{(a,b,c)} (z - c) = 0$$

$$2a(x - a) + 2b(y - b) + 5c(z - c) = 0$$

When $\mathbf{p} \neq (0, 0, 0) \in V$, the tangent space is given by an equation.

When $\mathbf{p} = (0, 0, 0) \in V$, the tangent space $0 = 0$!

Singularities or Singular points

When Implicit Function Theorem fails...

Singularities are the points of algebraic varieties in $\mathbb{C}^n$ where the general theory doesn't hold.

Smooth and Singular Points

Definition

Let $V \subset \mathbb{C}^n$ be defined by

$$f_1(z_1, \ldots, z_n) = \cdots = f_k(z_1, \ldots, z_n) = 0$$

and let the Jacobian matrix be

$$J(p) = \left(\frac{\partial f_i}{\partial z_j}(p) \right)$$

Then:

A point $p \in V$ is smooth (regular) if $\operatorname{rank} J(p) = n - \dim_p V$.

A point is singular if $\operatorname{rank} J(p) < n - \dim_p V$.

Dimension of an algebraic variety

Proposition - Definiton

Let V be an algebraic variety.

Let $p \in V$ be a nonsingular (smooth) point. Then

$$dim(V) = dim(T_pV)$$

In general, $dim(T_pV) \geq dim(V)$.

Isolated and Non-Isolated Singular Points

Let $V \subset \mathbb{C}^n$ be an algebraic variety.

A singular point $p \in V$ is called *isolated* if there exists a neighborhood U of p such that

$U \cap \mathrm{Sing}(V) = \{p\}$.

A singular point p is *non-isolated* if every neighborhood of p contains other singular points of V.

Example

Let $f(x, y, z) = y^2 - x^3 \in \mathbb{C}[x, y, z]$.

Consider the hypersurface $H = \{(x, y, z) \in \mathbb{C}^3 \mid y^2 - x^3 = 0\}$.

The singular set is $\{(0, 0, z) \mid z \in \mathbb{C}\}$.

Thus H has a **non-isolated singularity** since the singular locus is a whole line.

Blow up a point on a curve

Example

Consider $C := \{y^2 - x^3 - x^2 = 0\}$ in $\mathbb{C}^2$.

$(0,0) \in C$ is the only singular point. Put $y = ux$ in the equation:

$$(ux)^2 = x^2(x+1)$$

For all $x \neq 0$ we obtain x and y in terms of u as:

$$x = u^2 - 1, \quad y^2 = x^2 + x^3 = x^2(u^2-1)^2$$

Blow up a point on a curve

Example - continue

Consider the curve $C := \{y^2 - x^3 - x^2 = 0\}$ with $(0,0)$ the only singularity.

With $y = ux$, we get $x = u^2 - 1,\ y = u(u^2-1),\ z = u$ and

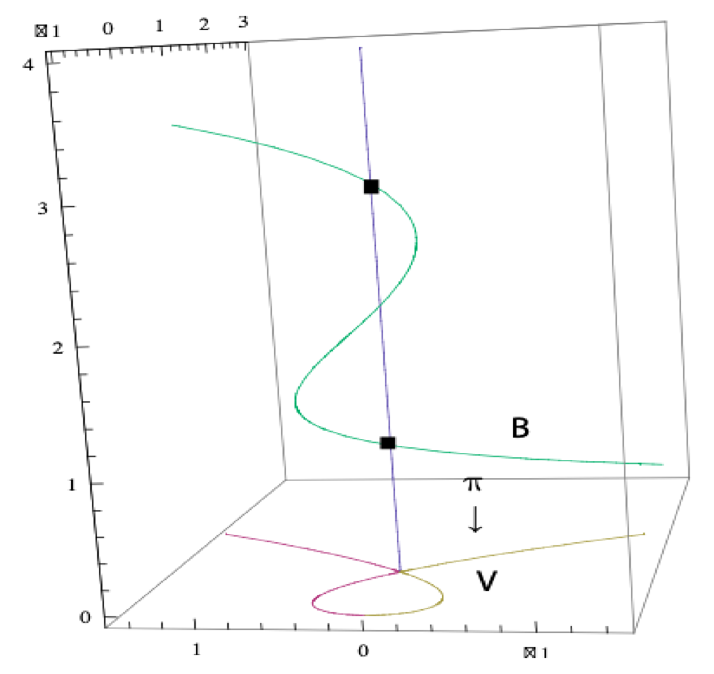
$$\{(x,y,u) \in \mathbb{C}^3 \mid x = u^2 - 1,\ y = u(u^2-1),\ z = u\}$$

which is a smooth curve and isomorphic to $y^2 - x^3 - x^2 = 0$ at every point except the origin.

Blow up a point on a curve

Example

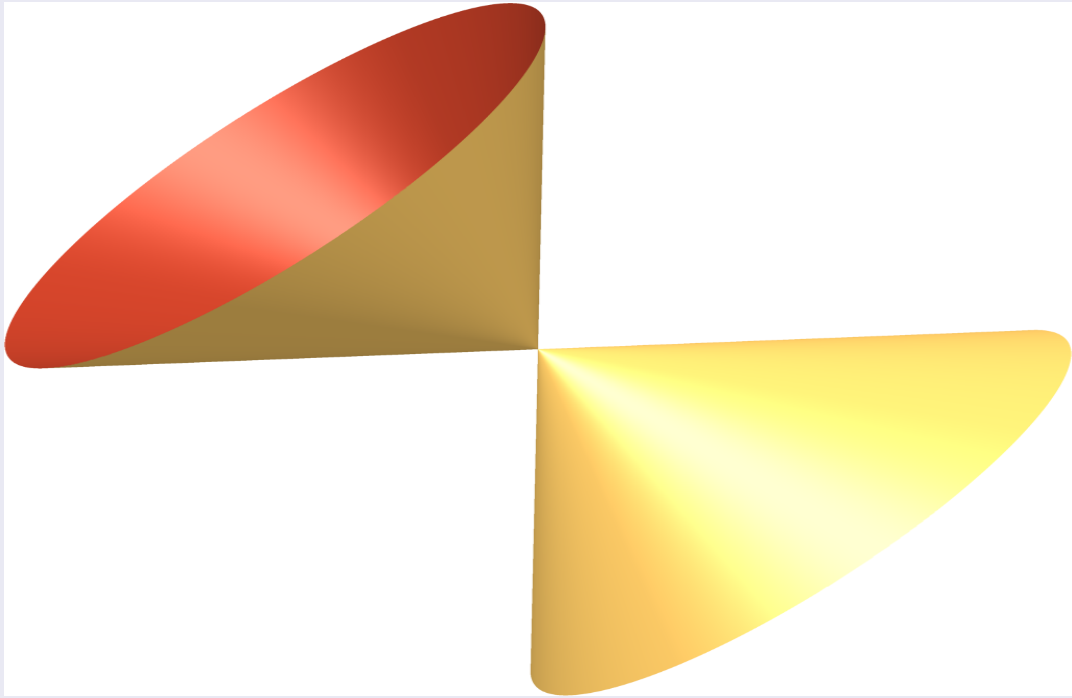
This new variety is a smooth curve isomorphic to $y^2 - x^3 - x^2 = 0$ at every point except the origin.



Blow up a point on a surface

Example

Let $V : z^2 - x^2 - y^2 = 0$



Blow up a point on a surface

Example

Let $V : z^2 - x^2 - y^2 = 0$ in $\mathbb{C}^3$.

Consider the parametrization $x = uz$ and $y = vz$. We have

$$\{(uz, vz, z, u, v) \in \mathbb{C}^5 \mid z^2u^2 + z^2v^2 - z^2 = 0\}$$

For $z \neq 0$, we get:

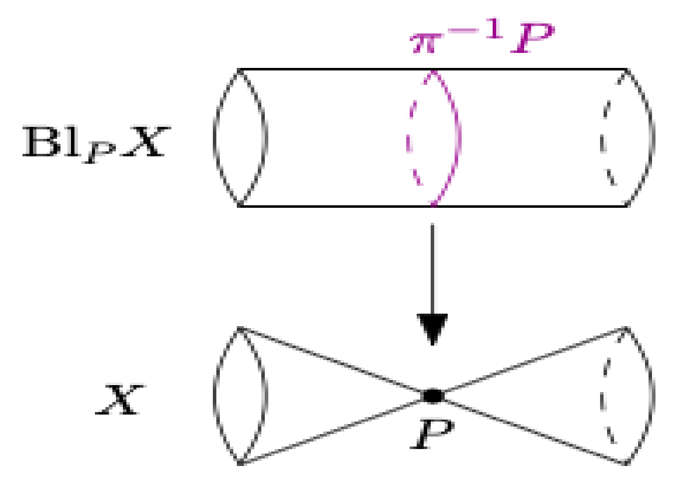
$$\{(zu, zv, z, u, v) \in \mathbb{C}^5 \mid u^2 + v^2 = 1, z \neq 0\}$$

Parametrization of a surface

Example - continue

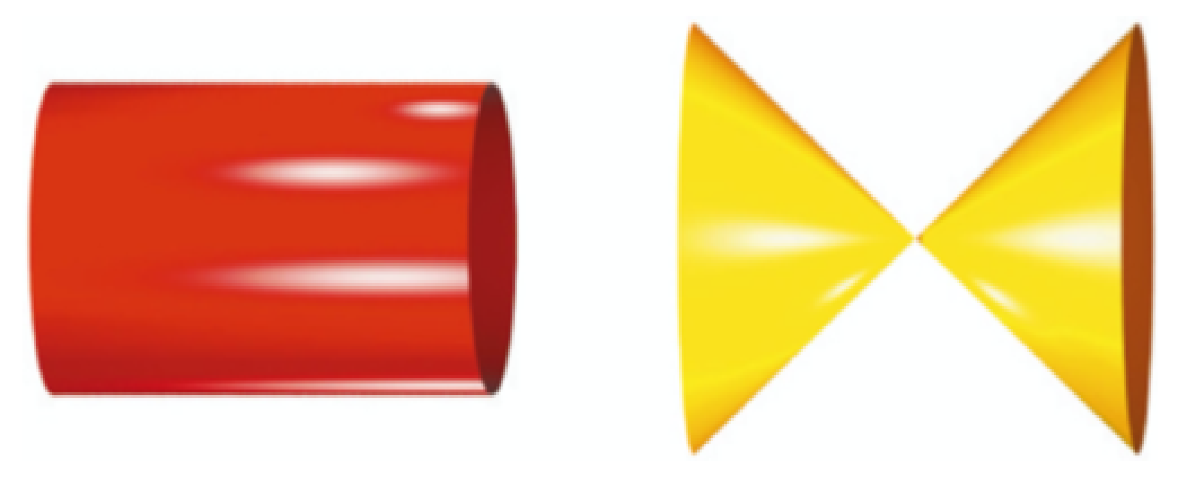
Resolution of the singularity $P = (0,0,0)$ is the cylinder

$$\{(z,u,v) \mid u^2 + v^2 = 1, z \neq 0\}$$



Blow up a point on a surface

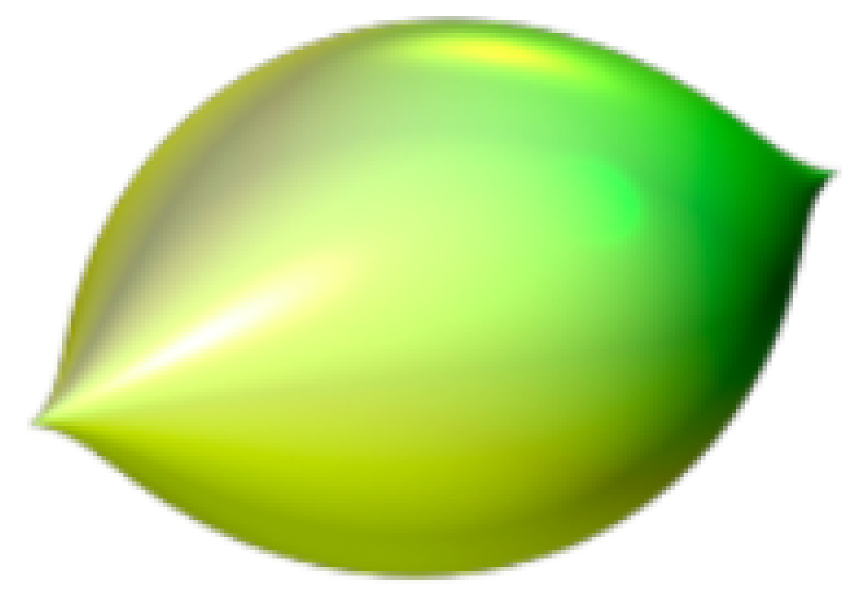
Example - continue



Blow up a point on a surface

Is it singular or smooth?

$$V : x^2 + z^2 - y^3(y-1)^3 = 0$$



Blow up - Resolution

The process of finding a good parametrization of a singular variety is called resolution of the singularity.

The parametrized variety should be nonsingular and isomorphic to our original variety almost everywhere.

Singular points - Resolution of Singularity

Definition

A resolution of singularities is a map $\pi : \tilde{V} \longrightarrow V$ such that $\tilde{V}$ smooth

(i) π is a **proper** surjective morphism,

(ii) the exceptional set $E := \pi^{-1}(\mathrm{Sing}(V))$ is a finite union of smooth codimension 1 subvarieties intersecting transversely,

(iii) π restricts to an isomorphism $\tilde{V} \setminus E \cong V \setminus \mathrm{Sing}(V)$

(iv) π can be obtained as a finite composition of blow-ups.

Singular points - Resolution of Singularity

Theorem (H. Hironaka, 1964)

Every algebraic variety V over a field of characteristic 0 admits a resolution of singularities.

Moreover, in dimension 2, there exists a minimal resolution which dominates all other resolutions, and it is unique up to isomorphism.

Singular points - Resolution of Singularity

In dimension ≥ 3, there is no canonical minimal resolution in general.

Instead, one studies:

- Minimal models (Mori theory),
- Crepant resolutions (when they exist).

The resolution is crepant if there is no discrepancy, meaning that in

$$K_Y = \pi^* K_X + \sum a_i E_i$$

all coefficients satisfy

$$a_i = 0$$

Classification of singularities

Simple singularities of surfaces (P.Du Val (1934))

The hypersurfaces in $\mathbb{C}^3$ are defined by one of the following equations

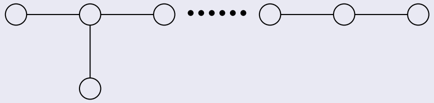
$$\begin{array}{ll} A_n & x^{n+1} + y^2 + z^2 = 0, \quad (n \geq 1) \\ D_n & x^{n-1} + xy^2 + z^2 = 0, \quad (n \geq 4) \\ E_6 & x^4 + y^3 + z^2 = 0 \\ E_7 & x^3y + y^3 + z^2 = 0 \\ E_8 & x^5 + y^3 + z^2 = 0 \end{array}$$

are called ADE or Du Val singularities.

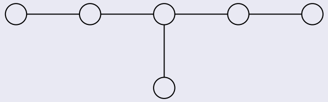
Minimal Resolution Graphs of Simple Singularities

A_n

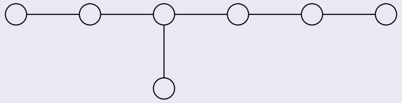
D_n



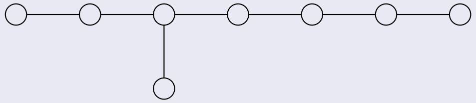
E_6



E_7



E_8



Classification of singularities

Simple Elliptic Singularities of surfaces (K.Saito (1974))

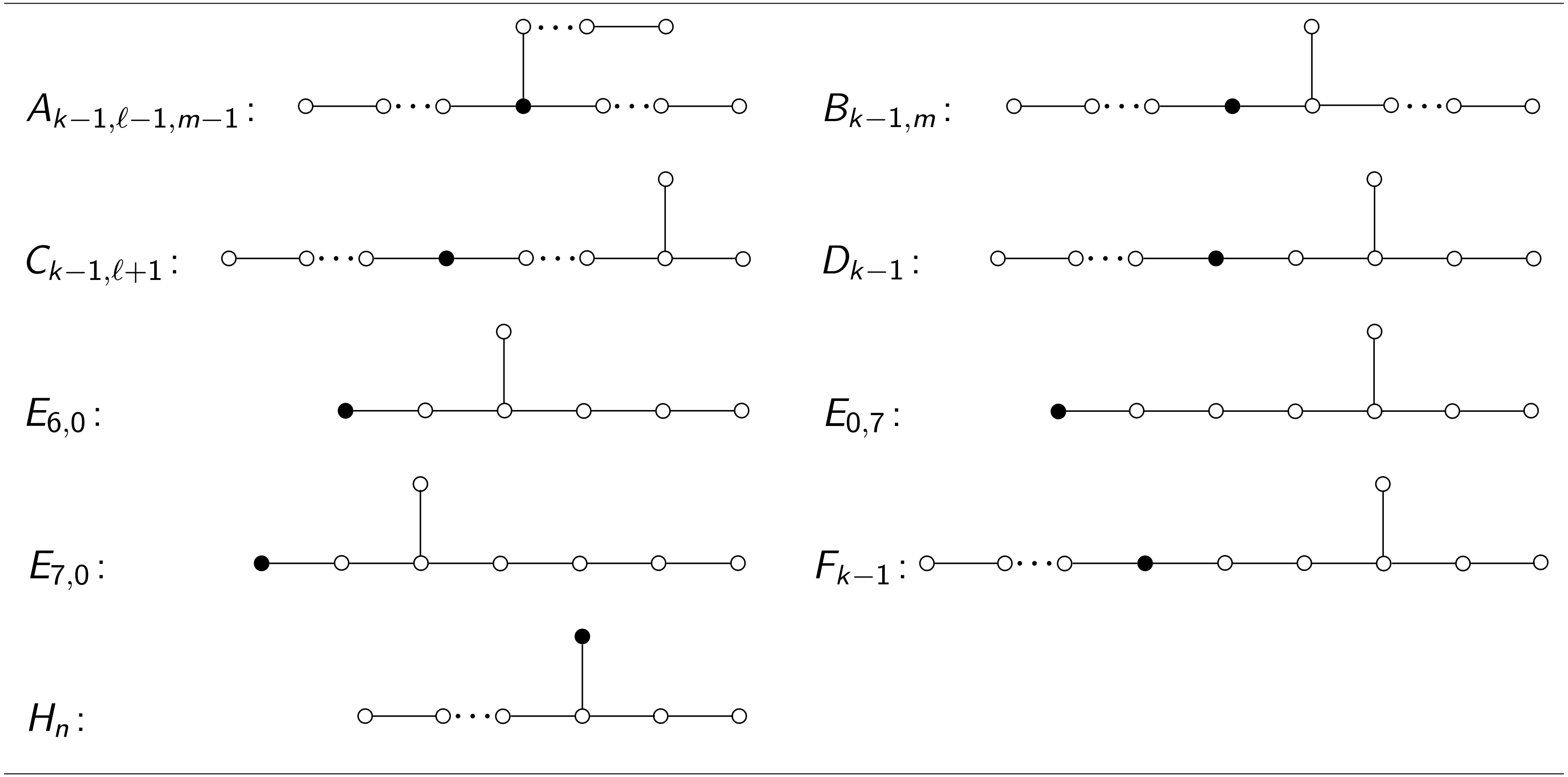
They are the hypersurfaces in $\mathbb{C}^3$ defined by one of the following equations:

$$\begin{array}{ll}
\tilde{E}_6 & x^6 + y^3 + z^2 + \lambda xyz = 0, \mathbf{E^2} = -\mathbf{1} \\
\tilde{E}_7 & x^4 + y^4 + z^2 + \lambda xyz = 0, \mathbf{E^2} = -\mathbf{2} \\
\tilde{E}_8 & x^3 + y^2 + \lambda xyz = 0, \mathbf{E^2} = -\mathbf{3} \\
\tilde{D}_5 & x^2 + y^2 + \lambda zw = 0, xy + z^2 + w^2 = 0, \mathbf{E^2} = -\mathbf{4}
\end{array}$$

where $\lambda \in \mathbb{C}$ such that these equations define an isolated singularity.

The exceptional divisor of the minimal resolution is is a nonsingular curve E with genus 1.

Surface singularities (M.Artin, 1966)



Surface Singularities and Finite Groups

A natural question is:

Which singularities arise as quotients by finite groups?

Surface Singularities and Finite Groups

Let $G \subset GL(2, \mathbb{C})$ be a finite subgroup acting linearly on $\mathbb{C}^2$.

Consider the quotient

$$X = \mathbb{C}^2/G$$

When it produce singularities?

Finite subgroups

Theorem (Klein, 1884)

A finite subgroup of $SL(2,\mathbb{C})$ is one of the following groups:

- The cyclic group $\mathbb{Z}/n\mathbb{Z}$ for $n > 1$.
- The binary dihedral group $\mathbb{B}D_{2n}$ for $n > 1$.
- The binary tetrahedral group $\mathbb{BT}$.
- The binary octahedral group $\mathbb{BO}$.
- The binary dodecahedral group $\mathbb{BD}$.

$SL(2,\mathbb{C})$ and singularities

Theorem (Klein, 1884)

Let $G \subset SL(2,\mathbb{C})$ be a finite subgroup acting linearly on $\mathbb{C}^2$.

Then the invariant ring $\mathbb{C}[x,y]^G$ is generated by

$$u, v, w \in \mathbb{C}[x,y]^G$$

satisfying a single relation

$$f(u,v,w) = 0$$

Equivalently,

$$\mathbb{C}[x,y]^G \cong \mathbb{C}[u,v,w]/(f)$$

so the quotient surface $\mathbb{C}^2/G$ is a hypersurface in $\mathbb{C}^3$.

$SL(2,\mathbb{C})$ and singularities - Example

Consider the cyclic group

$$\mathcal{C}_{n+1} = \langle g \rangle = \left\{ \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix} \middle| \, k = 0, 1, \ldots, n \right\} \subset SL(2,\mathbb{C})$$

where $\zeta = e^{\frac{2\pi i}{n+1}}$.

This group acts on $(u,v) \in \mathbb{C}^2$ by $(u,v) \mapsto \left(\zeta^k u, \zeta^{-k} v\right)$.

A monomial $u^a v^b$ transforms as $g \cdot \left(u^a v^b\right) = (\zeta u)^a \left(\zeta^{-1} v\right)^b = \zeta^{a-b} u^a v^b$.

So $u^a v^b$ is G-invariant iff $\zeta^{a-b} = 1 \quad \Longleftrightarrow \quad a - b \equiv 0 \pmod{n+1}$.

$SL(2,\mathbb{C})$ and singularities - Example

So $u^a v^b$ is G-invariant iff $\zeta^{a-b} = 1 \quad \Longleftrightarrow \quad a - b \equiv 0 \pmod{n+1}$.

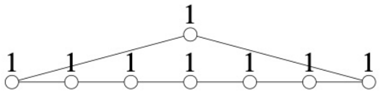
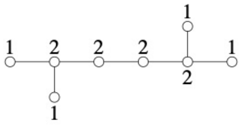
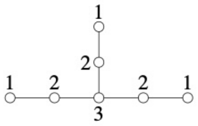
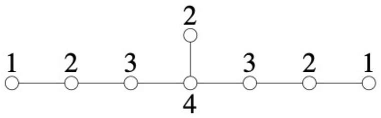
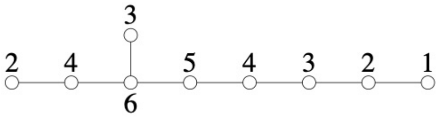
A generating set is $x = u^{n+1}, \quad y = v^{n+1}, \quad z = uv$.

These are invariant, and they satisfy the relation

$$xy = z^{n+1}$$

This is A_n singularities.

$SL(2,\mathbb{C})$ and singularities - Example

Finite subgroup of $SU(2)$		Affine simply laced Dynkin diagram	
$\mathbb{Z}/n\mathbb{Z}$	$\langle x \mid x^n = 1\rangle$	$\widetilde{A}_{n-1}$	
$\mathbb{B}D_{2n}$	$\langle x, y, z \mid x^2 = y^2 = y^n = xyz\rangle$	$\widetilde{D}_{n-2}$	
$\mathbb{BT}$	$\langle x, y, z \mid x^2 = y^3 = z^3 = xyz\rangle$	$\widetilde{E}_6$	
$\mathbb{BO}$	$\langle x, y, z \mid x^2 = y^3 = z^4 = xyz\rangle$	$\widetilde{E}_7$	
$\mathbb{BD}$	$\langle x, y, z \mid x^2 = y^3 = z^5 = xyz\rangle$	$\widetilde{E}_8$	

Irreducible representations of G and singularities

Definition

Let $\{\rho_0, \ldots, \rho_r\}$ be a complete set of pairwise non-isomorphic irreducible complex representations of G.

Let $\rho : G \to GL(V)$ be a fixed finite-dimensional representation of G.

The McKay graph $Q_\rho(G)$ is the oriented graph defined as

- The vertices are the irreducible representations $\rho_0, \ldots, \rho_r$.
- The number of arrows from ρ_i to ρ_j is m_{ij} where

$$\rho \otimes \rho_i \cong \bigoplus_{j=0}^{r} m_{ij}\rho_j$$

Properties of a McKay graph

- The multiplicities satisfy

$$m_{ij} \in \{0, 1\}$$

so, there are no multiple arrows.

- The matrix is symmetric

$$m_{ij} = m_{ji}$$

Thus the McKay graph is precisely an affine ADE diagram.

The McKay Correspondence

$$\begin{tikzcd}
\textbf{Finite subgroup } G \subset SU(2) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textbf{Irreducible representations of } G \arrow[d, Rightarrow] \\
\textbf{Quotient singularity } \mathbb{C}^2/G \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \textbf{McKay quiver } (m_{ij}) \arrow[d, Rightarrow] \\
\textbf{Minimal resolution} \arrow[r, Leftrightarrow] & \textbf{Affine ADE Dynkin diagram}
\end{tikzcd}$$

Finite subgroups of $\mathrm{GL}(2,\mathbb{C})$

Chevalley-Shephard-Todd Theorem, 1954

Let $G \subset GL(n,\mathbb{C})$ be a finite subgroup.

Then the following are equivalent:

(i) The quotient $\mathbb{C}^n/G$ is smooth.

(ii) The invariant ring $\mathbb{C}[x_1,\dots,x_n]^G$ is a polynomial ring.

(iii) The group G is generated by pseudo-reflections.

Finite subgroups of $\mathrm{GL}(2, \mathbb{C})$

Let $g \in GL(n, \mathbb{C})$ and $g \neq \mathrm{id}$.

For $A \in \mathrm{GL}(2, \mathbb{C})$, the following are equivalent:

1. A is a pseudo-reflection.
2. A is conjugate in $\mathrm{GL}(2, \mathbb{C})$ to a matrix of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad \lambda \neq 0, 1$$

3. The rank condition holds : $\mathrm{rank}(A - I) = 1$.

Finite subgroups of $\mathrm{GL}(2,\mathbb{C})$

Let $G \subset GL(2,\mathbb{C})$ be a finite subgroup.

Definition

The group G is called small if it contains no pseudo-reflections.

Theorem (Prill, 1967)

If G is small in $GL(2, C)$, then $\mathbb{C}^2/G$ is a surface with an isolated singularity at 0.

Small finite subgroups of $\mathrm{GL}(2, \mathbb{C})$

Using Brieskorn (1967) and Riemenschneider (1977), we get

Theorem (Brieskorn, 1967)

Any small finite subgroup of $\mathrm{GL}(2, \mathbb{C})$ is conjugate to one of the following groups:

(1) $\mathrm{C}_{n,q}$ with $0 < q < n$ and $(n, q) = 1$.

(2) $(\mu_{2m}, \mu_{2m}; \mathrm{BD}_q, \mathrm{BD}_q) \cong \langle 2, 2, q\rangle_m$ with $(m, 2) = 1$ and $(m, q) = 1$.

(3) $(\mu_{4m}, \mu_{2m}; \mathrm{BD}_q, \mathrm{C}_{2q})$ with $(m, 2) = 2$ and $(m, q) = 1$.

(4) $(\mu_{2m}, \mu_{2m}; \mathrm{BT}, \mathrm{BT}) \cong \langle 2, 3, 3\rangle_m$ with $(m, 6) = 1$.

(5) $(\mu_{6m}, \mu_{2m}; \mathrm{BT}, \mathrm{BD}_2)$ with $(m, 6) = 1$.

(6) $(\mu_{2m}, \mu_{2m}; \mathrm{BO}, \mathrm{BO}) \cong \langle 2, 3, 4\rangle_m$ with $(m, 6) = 1$.

(7) $(\mu_{2m}, \mu_{2m}; \mathrm{BI}, \mathrm{BI}) \cong \langle 2, 3, 5\rangle_m$ with $(m, 30) = 1$.

Generalized McKay Correspondence

$$\begin{tikzcd}
\textbf{Small finite subgroup } G \subset GL(2,\mathbb{C}) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textbf{Irreducible representations of } G \arrow[d, Rightarrow] \\
\textbf{Quotient singularity } \mathbb{C}^2/G \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \textbf{McKay graph } (m_{ij}) \arrow[d, Rightarrow] \\
\textbf{Minimal resolution} \arrow[r, Leftrightarrow] & \textbf{”certain” surface singularity}
\end{tikzcd}$$