Triple Integrals in Cartesian Coordinates

Change of Variables in Triple Integral

14.5 - 14.6

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14.5 Triple Integrals

For a bounded function $f(x,y,z)$ defined on a rectangular box B ($x_0 \le x \le x_1$, $y_0 \le y \le y_1$, $z_0 \le z \le z_1$), the triple integral of over B,

$$\iiint_B f(x,y,z)\,dV \quad \text{or} \quad \iiint_B f(x,y,z)\,dx\,dy\,dz$$

can be defined as a suitable limit of Rieman sums corresponding to partitions of B into subboxes by planes parallel to each of the coordinate planes.

All the properties of double integrals have analogues for triple integrals. In particular, a continuous function is integrable over a closed, bounded domain. If $f(x,y,z)=1$ on the domain D, then triple integral gives the volume of D;

$$\boxed{\text{Volume of } D = \iiint_D dV}$$

Some triple integrals can be evaluated by inspection, using symmetry and known volumes.

Example Evaluate $\displaystyle\iiint_{x^2+y^2+z^2 \le a^2} (2+x-\sin z)\,dV$

Solution

The domain of integration is the ball radius a centred at the origin. The integral of 2 over this ball is twice the ball's volume, that is, $8\pi a^3/3$. The integrals of x and $\sin z$ over the ball are both zero, since both functions are odd in one of the variables and the domain is symmetric about each coordinate plane.

Thus,

$$\iiint_{x^2+y^2+z^2 \le a^2} (2+x-\sin z)\,dV = \frac{8}{3}\pi a^3 + 0 + 0$$

$$= \frac{8}{3}\pi a^3.$$

14.6 Change of Variables in Triple Integrals

The change of variables formula for a double integral extends to triple (and higher order) integrals. Consider the transformation

$$x = x(u, v, w)$$
$$y = y(u, v, w)$$
$$z = z(u, v, w)$$

where x, y and z have continuous first partial derivatives wrt $u, v,$ and w. Near any point where the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ is nonzero, the transformation scales volume elements according to the formula

$$\boxed{dV = dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}$$

Thus, if the transformation is one-to-one from a domain S in uvw-space onto a domain D in xyz-space, and if

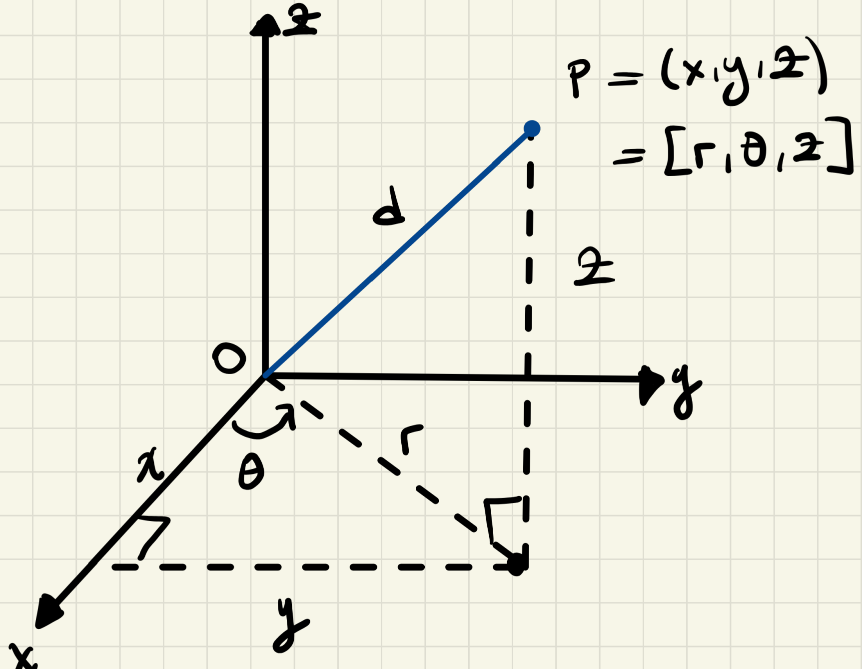
$$g(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w))$$

then

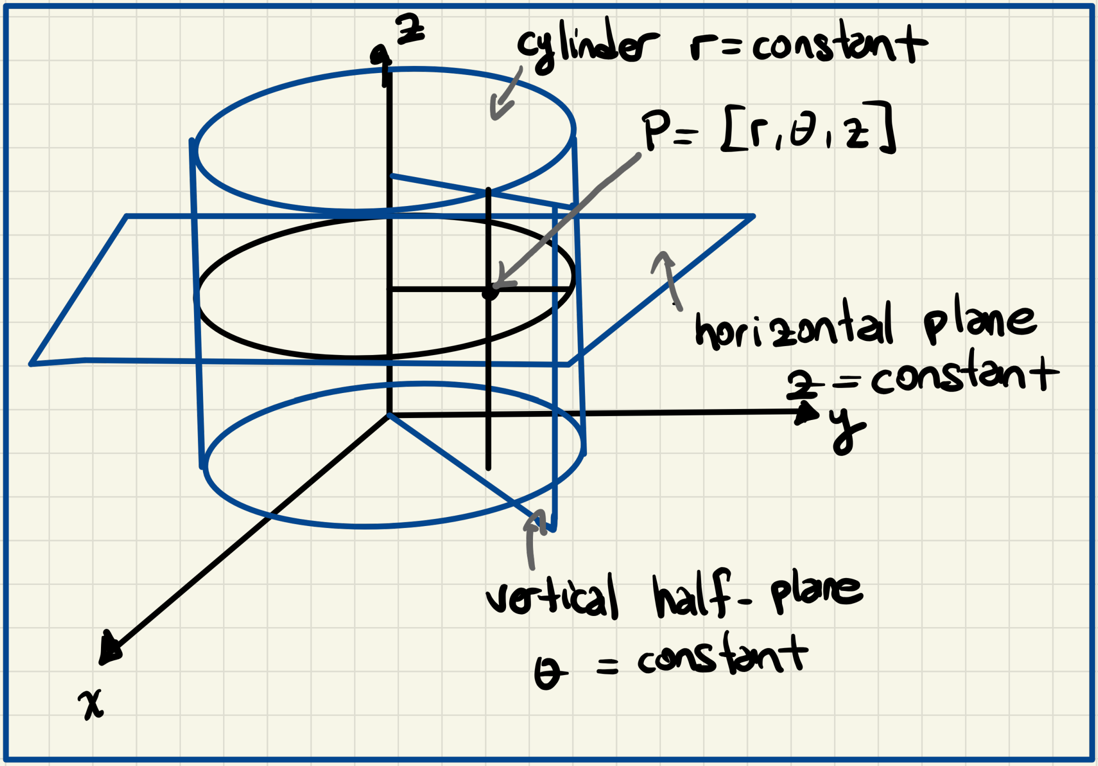
$$\boxed{\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_S g(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}$$

Cylindrical Coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$



(The cylindrical coordinates of a point)

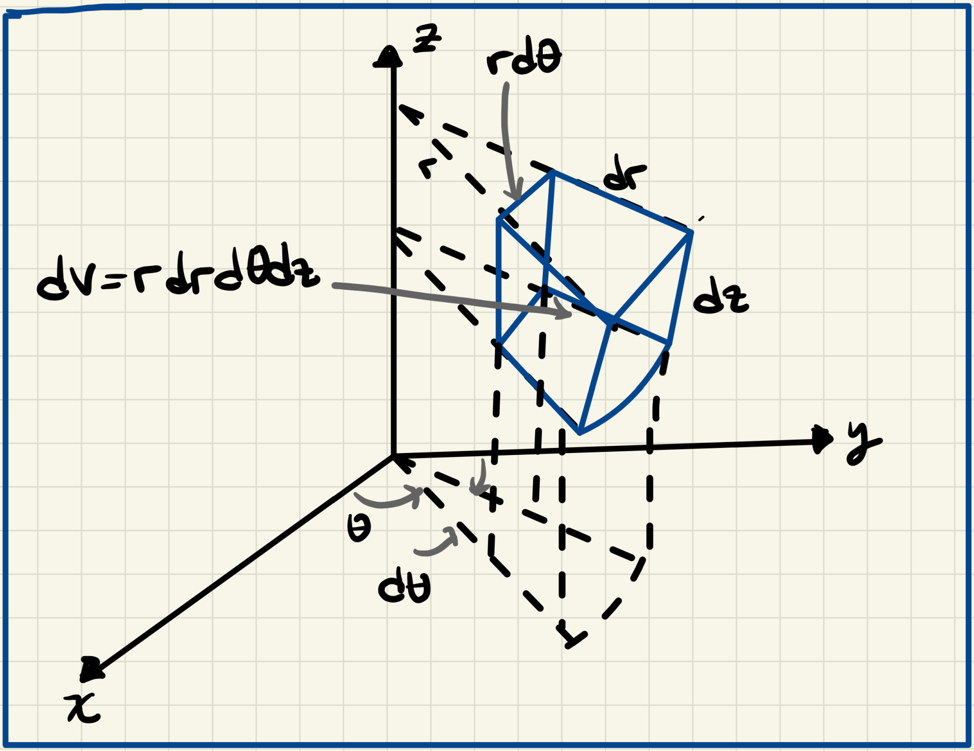


(The coordinate surfaces for cylindrical coordinates)

The volume element in cylindrical coordinates is

$$\boxed{dV = r\,dr\,d\theta\,dz}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$



Spherical Coordinates

$$x = R\sin\phi\cos\theta$$

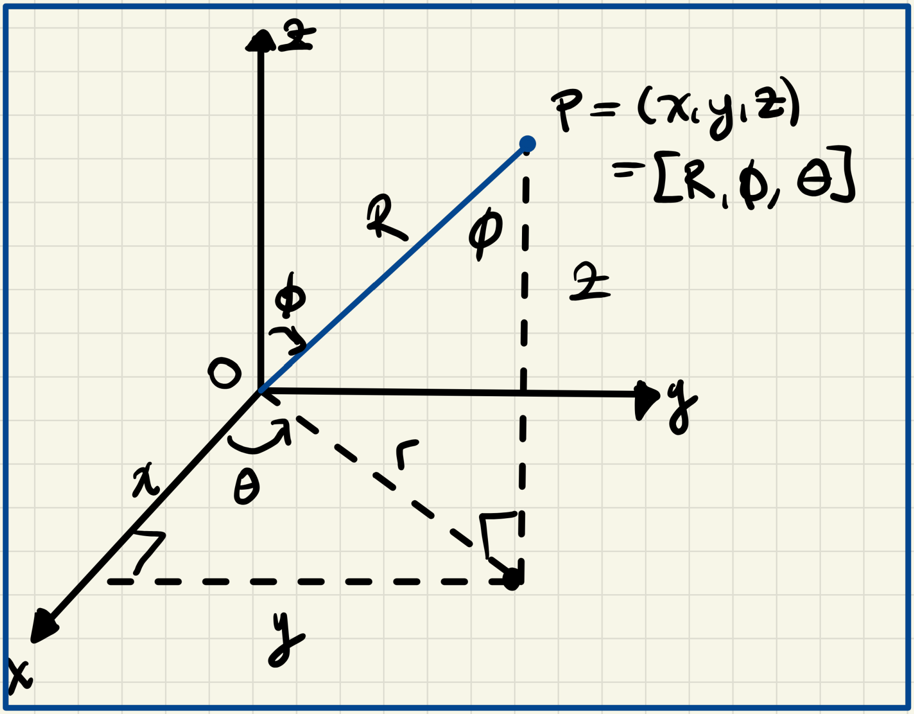
$$y = R\sin\phi\sin\theta$$

$$z = R\cos\phi$$

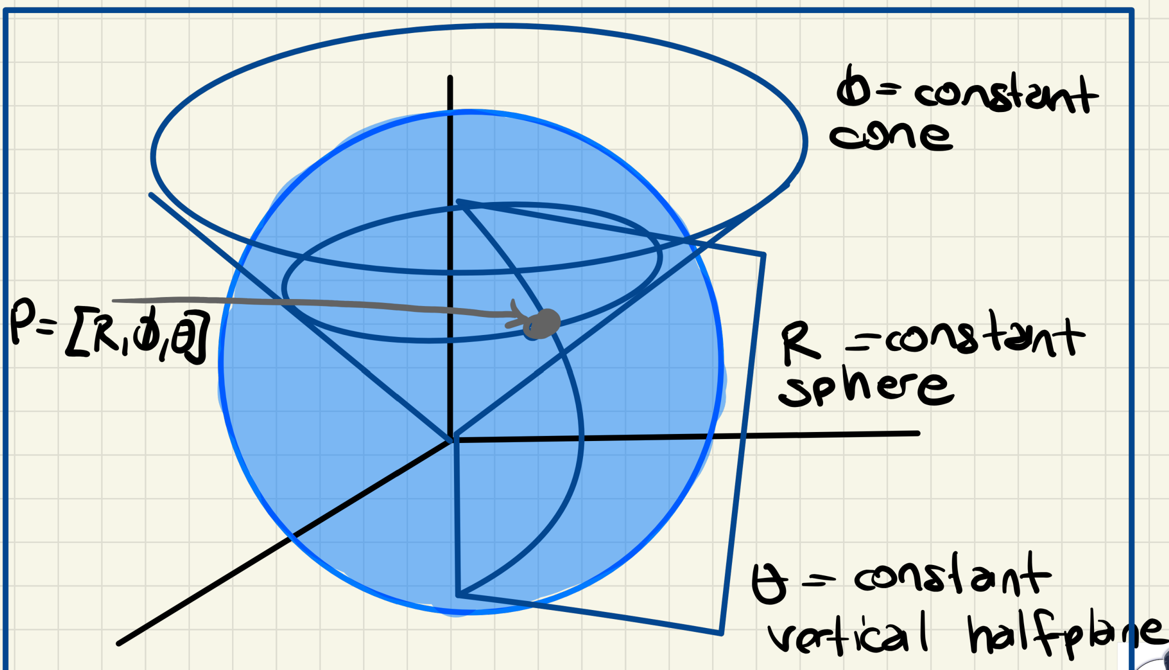
$$R^2 = x^2 + y^2 + z^2 = r^2 + z^2$$

$$r = \sqrt{x^2+y^2} = R\sin\phi$$

$$\tan\phi = r/z = \frac{\sqrt{x^2+y^2}}{z} \quad \text{and} \quad \tan\theta = \frac{y}{x}$$



(The spherical coordinates of a point)



The volume element in spherical coordinates is $\boxed{dV = R^2 \sin\phi \, dR \, d\phi \, d\theta}$