Double Integrals in Polar Coordinates & Change of Variables in Double Integrals

14.4

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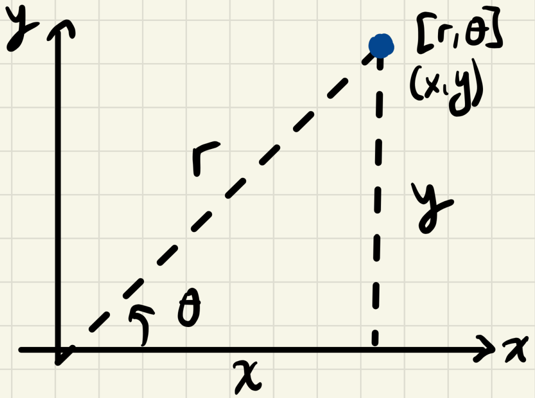
14.4 Double Integrals in Polar Coordinates

Recall that a point P with Ca esian coordinates (x, y) can also be located by its polar coordinates $[r, \theta]$, where r is the distance from P to the origin O, and θ is the angle OP makes with the positive direction of the x-axis.



$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$

$$y = r\sin\theta \qquad \tan\theta = y/x$$

(Polar - cartesian conversions)

Consider the problem of finding the volume V of the solid region lying above the xy-plane and beneath the paraboloid $z = 1 - x^2 - y^2$. Since the paraboloid intersects the xy-plane in the circle $x^2 + y^2 = 1$, the volume is given in Cartesian coordinates by

$$V = \iint_{x^2+y^2\le 1} (1-x^2-y^2)\,dA = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy$$

Evaluating this iterated integral would require considerable effort. However we can express the volume in terms of polar coordinates as

$$V = \iint_{r\le 1} (1-r^2)\,dA.$$

In transforming a double integral between Cartesian and polar coordinates, the area element transforms according to the formula

$$dx\,dy = dA = r\,dr\,d\theta$$

Change of Variables in Double Integrals

Suppose that x and y are expressed as functions of two other variablos u and v by the equations

$$x = x(u,v)$$
$$y = y(u,v)$$

We regard these equations as defining a transformation (or mapping) from points (u,v) in a uv - Cartesian plane to points (x,y) in the xy - plane. We say that the transformation is one-to-one from the set S in the uv-plane onto the set D in the xy - plane provided:

(i) every point in S gets mapped to a point in D.

(ii) every point in D is the image of a point in S, and

(iii) different points in S get mapped to different points in D.

If the transformation is one-to-one, the defining equations can be solved for u and v as functions of x and y, and the resulting inverse transformation,

$u = u(x, y)$

$v = v(x, y)$ is one-to-one from D onto S.

Let us assume that the functions $x(u, v)$ and $y(u, v)$ have continuous first partial derivatives and that the Jacobian determinant

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0 \text{ at } (u, v)$$

the Implicit Function Theorem implies that the transformation is one-to-one near (u,v) and the inverse transformation also has continuous first partial derivatives and nonzero Jacobian satisfying

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} \quad \text{on } D}$$

A one-to-one transformation can be used to transform the double integral

$\iint_D f(x,y)\, dA$ to a double integral over the corresponding set S in the uv-plane. Under the transformation, the integrand $f(x,y)$ becomes $g(u,v) = f(x(u,v), y(u,v))$.

If the value of u is fixed, say $u=c$, the equations $x=x(c,v)$ and $y=y(c,v)$ define a parametric curve (with v as parameter) in the xy-plane. This curve is called a u-curve corresponding to the value $u=c$. Similarly, for fixed $v=c$ the equations

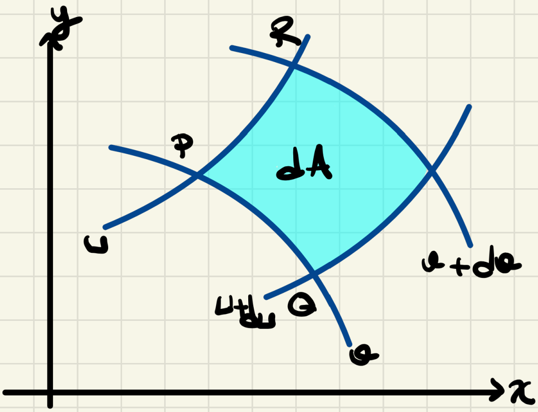
$$x=x(u,c) \quad \text{and} \quad y=y(u,c)$$

define a parametric curve (with parameter u) called a v-curve. Consider the differential area element bounded by the u-curves corresponding to nearby values u and $u+du$ and the v-curves corresponding to nearby values v and $v+dv$. Since these curves are smooth, for small values of du and dv the area element is approximately a parallelogram, and its area is approximately

$$dA = |\vec{PQ} \times \vec{PR}|$$

where P, Q and R are the points.

The error in this approximation becomes negligible compared with dA as du and dv approach zero.



Now $\overrightarrow{PQ} = dx\,i + dy\,j$, where

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

and $$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv.$$

However, $dv = 0$ along the v-curve PQ, so

$$\overrightarrow{PQ} = \frac{\partial x}{\partial u}du\,i + \frac{\partial y}{\partial u}du\,j$$

Similarly,

$$\overrightarrow{PR} = \frac{\partial x}{\partial v}dv\,i + \frac{\partial y}{\partial v}dv\,j.$$

Hence,

$$dA = \left| \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du & 0 \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv & 0 \end{vmatrix} \right| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$dA = dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Theorem Change of variables formula for double integrals

Let $x = x(u,v)$, $y = y(u,v)$ be a one-to-one transformation from a domain S in the uv-plane onto a domain D in the xy-plane. Suppose that the functions x and y, and their first partial derivatives wrt u and v, are continuous in S. If $f(x,y)$ is integrable on D, and if $g(u,v) = f(x(u,v), y(u,v))$, then g is integrable on S and

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_S g(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv}$$