Questions of December

Solutions

12 January, 2014

Solution of the Question 1

Let $deg(A)$ denote the number of roads to the city A. Since we have $0 \leq deg(A) \leq 19$ for all A, the set $\{deg(A) : A \in Country\}$ has the maximum element. Let A_0 be the city such that $deg(A_0)$ is maximum. Now denote the cities connected to A_0 by a road as $B_1, B_2, ..., B_k$ and denote the others as $C_1, C_2, ..., C_{19-k}$. Now assume that there is no triangle. Then since B_i's are connected to A_0, B_i and B_j must not be connected for $i \neq j$ since otherwise we would have a triangle. Let $S_1 = \{A_0, B_1, B_2, ...B_k\}$ and $S_2 = \{C_1, C_2, ..., C_{19-k}\}$. Then in S_1 there are exactly k roads. Now consider other $101 - k$ roads. All these $101 - k$ roads have at least one end in S_2 since otherwise S_1 contains a triangle. So we have at most

$$\sum_{i=1}^{19-k} deg(C_i)$$

roads in this country. Since $deg(C_i) \leq k = deg(A_0)$ for all $i = 1, 2, ..., 19 - k$ we get

$$101 \leq k + \sum_{i=1}^{19-k} deg(C_i) \leq k + (19 - k) * k = 20k - k^2$$

hence we have

$$20k - k^2 \geq 101$$

or equivalently

$$(k - 10)^2 + 1 \leq 0$$

which is a contradiction. Hence, we are done.

Solution of the Question 2

Suppose that all points are not on the same line. Then there are some points $P_1, P_2, ..., P_k$ and some lines $l_1, l_2, ..., l_m$ such that $|P_i l_j| \neq 0$. Since there are finitely many points, there is a point P_i and a line l_j such that $|P_i l_j|$ is minimum (that is we choose a line and a point such that there are no other points and lines closer to each other). Now choose a point P on l_j and join it to P_i. There is no way to place a third point on that line (Why?).