Questions of December

Solutions

12 January, 2014

Solution of the Question 1

Let \( \text{deg}(A) \) denote the number of roads to the city \( A \). Since we have \( 0 \leq \text{deg}(A) \leq 19 \) for all \( A \), the set \( \{ \text{deg}(A) : A \in \text{Country} \} \) has the maximum element. Let \( A_0 \) be the city such that \( \text{deg}(A_0) \) is maximum. Now denote the cities connected to \( A_0 \) by a road as \( B_1, B_2, ..., B_k \) and denote the others as \( C_1, C_2, ..., C_{19-k} \). Now assume that there is no triangle. Then since \( B_i \)'s are connected to \( A_0 \), \( B_i \) and \( B_j \) must not be connected for \( i \neq j \) since otherwise we would have a triangle. Let \( S_1 = \{ A_0, B_1, B_2, ...B_k \} \) and \( S_2 = \{ C_1, C_2, ..., C_{19-k} \} \). Then in \( S_1 \) there are exactly \( k \) roads. Now consider other \( 101 - k \) roads. All these \( 101 - k \) roads have at least one end in \( S_2 \) since otherwise \( S_1 \) contains a triangle. So we have at most

\[
\sum_{i=1}^{19-k} \text{deg}(C_i)
\]

roads in this country. Since \( \text{deg}(C_i) \leq k = \text{deg}(A_0) \) for all \( i = 1, 2, ..., 19 - k \) we get

\[
101 \leq k + \sum_{i=1}^{19-k} \text{deg}(C_i) \leq k + (19 - k) \times k = 20k - k^2
\]

hence we have

\[
20k - k^2 \geq 101
\]

or equivalently

\[
(k - 10)^2 + 1 \leq 0
\]

which is a contradiction. Hence, we are done.

Solution of the Question 2

Suppose that all points are not on the same line. Then there are some points \( P_1, P_2, ..., P_k \) and some lines \( l_1, l_2, ..., l_m \) such that \( |P_i l_j| \neq 0 \). Since there are finitely many points, there is a point \( P_i \) and a line \( l_j \) such that \( |P_i l_j| \) is minimum (that is we choose a line and a point such that there are no other points and lines closer to each other). Now choose a point \( P \) on \( l_j \) and join it to \( P_i \). There is no way to place a third point on that line (Why?).